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FINAL YEAR B.Sc. DEGREE EXAMINATION, MARCH/APRIL 2005

Part III—Group I—Mathematics

Paper III—ALGEBRA

Time : Three Hours

Maximum : 65 Marks

Maximum marks for each unit is 13.

Unit I

1. Define a relation \mathcal{R} on z by setting $n\mathcal{R}m$ if and only if $nm \geq 0$. Verify whether \mathcal{R} is an equivalence relation on z . (3 marks)
2. Define $*$ on Q^+ by $a * b = \frac{ab}{2}$. Prove that $(Q^+, *)$ is a group. (4 marks)
3. Let $a \in G$, G a group. Prove that $H = \{a^n : n \in z\}$ is a subgroup of G and that it is the smallest subgroup of G that contains a . (4 marks)

4. Prove that every group is isomorphic to a group of permutations. (5 marks)
5. Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$ as a product of transpositions. (4 marks)

Unit II

6. Show that every group of prime order is cyclic. (5 marks)
7. Compute the factor group $z_4 \times z_6 / \langle (0, 1) \rangle$. Verify whether it is isomorphic to z_4 . (5 marks)
8. If H is a subgroup of G and N is a normal subgroup of G show that $\frac{HN}{N} \cong H/(H \cap N)$. (5 marks)
9. Let X be a G -set. Define the isotropy subgroup G_x of $x \in X$. Prove that G_x is a subgroup of G for each $x \in X$. (5 marks)

Unit III

10. Prove that z_n under addition modulo n and multiplication modulo n form a ring. Verify whether it is an integral domain. (5 marks)
11. Define the characteristic of a ring R . Prove that if R is a ring with unity then R has characteristic $n > 0$ if and only if n is the smallest positive integer such that $n \cdot 1 = 0$. (5 marks)
12. Show that the quaternions form a skew field under addition and multiplication. (5 marks)
13. Prove that any two fields of quotients of an integral domain are isomorphic. (5 marks)

Unit IV

14. Let A, B be ideals of a ring R . Define
 $A + B = \{a + b/a \in A, b \in B\}$.
 Show that $A + B$ is an ideal of R . (5 marks)
15. Define a prime ideal of a ring. Prove that an ideal $N \neq R$ is prime if and only if $\frac{R}{N}$ is an integral domain where R is a commutative ring with unity. (5 marks)
16. Prove that a non-zero polynomial $f(x) \in F[x]$ of degree n can have at most n zeroes in a field F . (5 marks)
17. If F is a field prove that every non-constant polynomial $f(x) \in F[x]$ can be expressed uniquely as a product of irreducible polynomials. (5 marks)

Unit V

18. Let V be a vector space over F and F^1 be a subfield of F . Show that V is a vector space over F^1 also. (5 marks)
19. Show that if V_1 is a subspace of V_2 and V_2 is a subspace of V , then V_1 is a subspace of V . (5 marks)
20. Prove that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2),$$
 where V_1 and V_2 are subspaces of a vector space V . (5 marks)
21. Prove that if V, V^1 are vector spaces over a field F then the set of all linear transformations of V to V^1 form a vector space over F . (5 marks)